

Sharp pointwise estimates for functions in the Sobolev spaces $H^s(\mathbb{R}^n)$

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Abstract

We provide the optimal value of the constant $K(n, m)$ in the Gagliardo-Nirenberg supnorm inequality $\|u\|_{L^\infty(\mathbb{R}^n)} \leq K(n, m) \|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2m}} \|D^m u\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2m}}$, $m > n/2$, and its generalizations to the Sobolev spaces $H^s(\mathbb{R}^n)$ of arbitrary order $s > n/2$ as well.

1. Introduction

In recent decades there has been a growing interest in determining the sharpest form of many important inequalities in analysis, see e.g. [1, 2, 3, 4, 5, 8, 10] and references therein. A noticeable miss is the fundamental Gagliardo-Nirenberg supnorm inequality

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq K(n, m) \|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2m}} \|D^m u\|_{L^2(\mathbb{R}^n)}^{\frac{n}{2m}} \quad (1.1)$$

for functions $u \in H^m(\mathbb{R}^n)$ when $m > n/2$ (see [7, 9]), as well as some of its generalizations. Here, as usual, $H^m(\mathbb{R}^n)$ is the Sobolev space of functions $u \in L^2(\mathbb{R}^n)$ with all derivatives of order up to m in $L^2(\mathbb{R}^n)$, which is a Banach space under its natural norm defined by

$$\|u\|_{H^m(\mathbb{R}^n)} = \left\{ \|u\|_{L^2(\mathbb{R}^n)}^2 + \|D^m u\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/2}, \quad (1.2a)$$

where

$$\|D^m u\|_{L^2(\mathbb{R}^n)} = \left\{ \sum_{i_1=1}^n \sum_{i_2=1}^n \dots \sum_{i_m=1}^n \|D_{i_1} D_{i_2} \dots D_{i_m} u\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/2} \quad (1.2b)$$

(with D_i denoting the weak derivative with respect to the variable x_i). In this brief note, we will review some basic results in order to derive the optimal (i.e., minimal) value for the constant $K(m, n)$ in (1.1) above. It will be seen in Section 2 that it turns out to be

$$K(n, m) = \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma(\frac{n}{2m})}{\sigma(\frac{n}{2m})} \right\}^{-\frac{1}{2}} \left\{ \frac{n}{2m-n} \right\}^{-\frac{n}{4m}} \left\{ \frac{2m}{2m-n} \right\}^{\frac{1}{2}} \quad (1.3)$$

where $\sigma(r) := r\pi$ and $\Gamma(\cdot)$ is the Gamma function (for its definition, see e.g. [6], p. 7). For example, we get, with $m = 2$ and $n = 1, 2, 3$, the sharp pointwise estimates

$$\|u\|_{L^\infty(\mathbb{R})} \leq \frac{\sqrt[4]{2}}{\sqrt[8]{27}} \|u\|_{L^2(\mathbb{R})}^{\frac{3}{4}} \|D^2 u\|_{L^2(\mathbb{R})}^{\frac{1}{4}}, \quad (1.4a)$$

$$\|u\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{2} \|u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}} \|D^2 u\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \quad (1.4b)$$

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq \frac{\sqrt[8]{12}}{\sqrt{6\pi}} \|u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \|D^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{3}{4}}, \quad (1.4c)$$

and so forth. (In [10], it is obtained that $\|u\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{\sqrt{2\pi}} \|Du\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}} \|D^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2}}$, which is also shown to be optimal.) Also, setting

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)} := \left\{ \int_{\mathbb{R}^n} |\xi|^{2s} |\hat{u}(\xi)|^2 d\xi \right\}^{\frac{1}{2}} \quad (1.5)$$

for real $s > 0$,¹ where $\hat{u}(\cdot)$ denotes the Fourier transform of $u(\cdot)$, that is,

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) dx, \quad \xi \in \mathbb{R}^n, \quad (1.6)$$

and letting $H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|_{\dot{H}^s(\mathbb{R}^n)} < \infty\}$ be the Sobolev space of order s , we get the following generalization of (1.1), (1.3) above. If $s > n/2$, then

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq K(n, s) \|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} \quad (1.7a)$$

for all $u \in H^s(\mathbb{R}^n)$, with the optimal value of the constant $K(n, s)$ being given by

$$K(n, s) = \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma(\frac{n}{2s})}{\sigma(\frac{n}{2s})} \right\}^{-\frac{1}{2}} \left\{ \frac{n}{2s-n} \right\}^{-\frac{n}{4s}} \left\{ \frac{2s}{2s-n} \right\}^{\frac{1}{2}} \quad (1.7b)$$

for any $s > n/2$, where, as before, $\sigma(r) := r\pi$. The proof of (1.1), (1.3), (1.7) and of the sharpness of the values for K given in (1.3), (1.7b) above is provided in the sequel; in addition, a second classical estimate for $\|u\|_{L^\infty(\mathbb{R}^n)}$ when $u \in H^s(\mathbb{R}^n)$, $s > n/2$, is also reexamined here (see (2.4) below), so as to be similarly presented in its sharpest form.

¹Note that (1.5) corresponds to (1.2b) when $s = m$ (m integral), that is: $\|u\|_{\dot{H}^m(\mathbb{R}^n)} = \|D^m u\|_{L^2(\mathbb{R}^n)}$.

2. Proof of (1.1), (1.3), (1.7) and other optimal supnorm results in $H^s(\mathbb{R}^n)$

To obtain the results stated in Section 1, we first review the following basic lemma. We recall that \hat{u} denotes the Fourier transform of u , cf. (1.6) above.

Lemma 2.1. *Let $u \in L^2(\mathbb{R}^n)$. If $\hat{u} \in L^1(\mathbb{R}^n)$, then $u \in L^\infty(\mathbb{R}^n)$ and*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq (2\pi)^{-n/2} \|\hat{u}\|_{L^1(\mathbb{R}^n)}. \quad (2.1)$$

Moreover, equality holds in (2.1) if \hat{u} is real-valued and of constant sign (say, nonnegative).

Proof: Clearly, (2.1) is valid if $u \in \mathcal{S}(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of smooth, rapidly decreasing functions at infinity ([6], p. 4), since we have, in this case, the representation (see e.g. [6], p. 16)

$$u_m(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{u}_m(\xi) d\xi, \quad \forall x \in \mathbb{R}^n. \quad (2.2)$$

For general $u \in L^2(\mathbb{R}^n)$ with $\hat{u} \in L^1(\mathbb{R}^n)$, let $\{\hat{u}_m\}$ be a sequence of Schwartz approximants to $\hat{u} \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $\|\hat{u}_m - \hat{u}\|_{L^1(\mathbb{R}^n)} \rightarrow 0$ and $\|\hat{u}_m - \hat{u}\|_{L^2(\mathbb{R}^n)} \rightarrow 0$ as $m \rightarrow \infty$, and let $u_m \in \mathcal{S}(\mathbb{R}^n)$ be the Fourier inverse of \hat{u}_m , for each m . Applying (2.1) to $\{u_m\}$, we see that $\{u_m\}$ is Cauchy in $L^\infty(\mathbb{R}^n)$, so that $\|u_m - v\|_{L^\infty(\mathbb{R}^n)} \rightarrow 0$ for some $v \in L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$. Since we have $\|u_m - u\|_{L^2(\mathbb{R}^n)} \rightarrow 0$, it follows that $u = v$. This shows that $u \in L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ and, letting $m \rightarrow \infty$ in (2.2), we also have

$$u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \hat{u}(\xi) d\xi, \quad \forall x \in \mathbb{R}^n, \quad (2.2')$$

from which (2.1) immediately follows. In particular, in the event that $\hat{u}(\xi) \geq 0$ for all ξ , we get from (2.2') that $u(0) = (2\pi)^{-n/2} \|\hat{u}\|_{L^1(\mathbb{R}^n)}$, so that (2.1) becomes an identity in this case. \square

We observe that the hypotheses of Lemma 2.1 are satisfied for $u \in H^s(\mathbb{R}^n)$ if $s > n/2$. An important consequence of this fact is the fundamental embedding property revisited next, where the norm in $H^s(\mathbb{R}^n)$ is set to be (in accordance with (1.2a) above):

$$\|u\|_{H^s(\mathbb{R}^n)} = \left\{ \|u\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 \right\}^{1/2} = \left\{ \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi \right\}^{1/2}. \quad (2.3)$$

Theorem 2.1. *Let $s > n/2$. If $u \in H^s(\mathbb{R}^n)$, then $u \in L^\infty(\mathbb{R}^n) \cap C^0(\mathbb{R}^n)$ and*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma(\frac{n}{2s})}{\sigma(\frac{n}{2s})} \right\}^{-\frac{1}{2}} \cdot \|u\|_{H^s(\mathbb{R}^n)}, \quad (2.4)$$

where $\sigma(r) = r\pi$. Moreover, equality holds in (2.4) when $\hat{u}(\xi) = c/(1 + |\xi|^{2s}) \forall \xi \in \mathbb{R}^n$ for some constant $c \geq 0$, so that the constant given in (2.4) above is optimal.

Proof: Let $u \in H^s(\mathbb{R}^n)$, with $s > n/2$. By (2.1) and Cauchy-Schwarz's inequality, we have

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq (2\pi)^{-n/2} \int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1/2} (1 + |\xi|^{2s})^{1/2} |\hat{u}(\xi)| d\xi \quad (2.5a)$$

$$\leq (2\pi)^{-n/2} \left\{ \int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1} d\xi \right\}^{1/2} \left\{ \int_{\mathbb{R}^n} (1 + |\xi|^{2s}) |\hat{u}(\xi)|^2 d\xi \right\}^{1/2} \quad (2.5b)$$

$$= \{4\pi\}^{-n/4} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-1/2} \left\{ \frac{\sin \sigma(\frac{n}{2s})}{\sigma(\frac{n}{2s})} \right\}^{-1/2} \cdot \|u\|_{H^s(\mathbb{R}^n)}$$

by (2.3), since, using polar coordinates and the change of variable $t = (1 + r^{2s})^{-1}$ we obtain

$$\int_{\mathbb{R}^n} (1 + |\xi|^{2s})^{-1} d\xi = \omega_n \int_0^\infty (1 + r^{2s})^{-1} r^{n-1} dr = \frac{\omega_n}{2s} \int_0^1 t^{1-\frac{n}{2s}} (1-t)^{\frac{n}{2s}-1} dt = \frac{\omega_n}{n} \frac{\sigma(\frac{n}{2s})}{\sin \sigma(\frac{n}{2s})}$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit ball in \mathbb{R}^n (see [6], p. 8), and $\sigma(r) = r\pi$. This shows (2.4). Finally, if $\hat{u}(\xi) = c/(1 + |\xi|^{2s})$ for all ξ , for some $c \geq 0$ constant, then equality holds in both steps (2.5a) and (2.5b) above, so that (2.4) is an identity in this case, as claimed. \square

We are now in very good standing to obtain (1.1), (1.7a) with their sharpest constants.

Theorem 2.2. *Let $s > n/2$. If $u \in H^s(\mathbb{R}^n)$, then*

$$\|u\|_{L^\infty(\mathbb{R}^n)} \leq K(n, s) \|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} \quad (2.6)$$

with $K(n, s)$ defined in (1.7b). Moreover, equality holds in (2.6) if $\hat{u}(\xi) = c/(1 + |\xi|^{2s})$ $\forall \xi \in \mathbb{R}^n$, $c \geq 0$ constant, so that the numerical value of K given in (1.7b) is optimal.

Proof: Let $s > n/2$, $u \in H^s(\mathbb{R}^n)$ fixed. Given $\lambda > 0$, setting $u_\lambda \in H^s(\mathbb{R}^n)$ by $u_\lambda(x) := u(\lambda x)$, we have, by (2.4), Theorem 2.1,

$$\begin{aligned} \|u\|_{L^\infty(\mathbb{R}^n)} &= \|u_\lambda\|_{L^\infty(\mathbb{R}^n)} \leq \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma(\frac{n}{2s})}{\sigma(\frac{n}{2s})} \right\}^{-\frac{1}{2}} \cdot \|u_\lambda\|_{H^s(\mathbb{R}^n)} \\ &= \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma(\frac{n}{2s})}{\sigma(\frac{n}{2s})} \right\}^{-\frac{1}{2}} \cdot \left\{ \lambda^{-n} \|u\|_{L^2(\mathbb{R}^n)}^2 + \lambda^{2s-n} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 \right\}^{\frac{1}{2}} \end{aligned}$$

for $\lambda > 0$ arbitrary. Choosing λ that minimizes the last term on the right of the above expression gives us (2.6), with $K(n, s)$ defined by (1.7b), as claimed. Now, to show that the value provided in (1.7b) is the best possible, we proceed as follows. First, we observe that, by Young's inequality,

$$\left\{ \frac{n}{2s-n} \right\}^{-\frac{n}{4s}} \left\{ \frac{2s}{2s-n} \right\}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^n)}^{1-\frac{n}{2s}} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} \leq \left\{ \|u\|_{L^2(\mathbb{R}^n)}^2 + \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 \right\}^{\frac{1}{2}} = \|u\|_{H^s(\mathbb{R}^n)} \quad (2.7)$$

for all $u \in H^s(\mathbb{R}^n)$. Therefore, taking $w \in H^s(\mathbb{R}^n)$ defined by $\widehat{w}(\xi) = c/(1 + |\xi|^{2s})$, $c \geq 0$, we get, with $K(n, s)$ given in (1.7b):

$$\begin{aligned} \|w\|_{L^\infty(\mathbb{R}^n)} &\leq K(n, s) \|w\|_{L^2(\mathbb{R}^n)}^{1 - \frac{n}{2s}} \|w\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}} && [\text{by (2.6)}] \\ &\leq \{4\pi\}^{-\frac{n}{4}} \left\{ \frac{n}{2} \Gamma\left(\frac{n}{2}\right) \right\}^{-\frac{1}{2}} \left\{ \frac{\sin \sigma(\frac{n}{2s})}{\sigma(\frac{n}{2s})} \right\}^{-\frac{1}{2}} \cdot \|w\|_{H^s(\mathbb{R}^n)} && [\text{by (1.7b), (2.7)}] \\ &= \|w\|_{L^\infty(\mathbb{R}^n)}, && [\text{by Theorem 2.1}] \end{aligned}$$

thus showing that we have $\|w\|_{L^\infty(\mathbb{R}^n)} = K(n, s) \|w\|_{L^2(\mathbb{R}^n)}^{1 - \frac{n}{2s}} \|w\|_{\dot{H}^s(\mathbb{R}^n)}^{\frac{n}{2s}}$, as claimed. \square

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